Seventh-Graders' Mathematical Modelling on Completion of a Three-Year Program

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This paper addresses 7th-grade children's mathematical modelling at the end of a 3-year program. The children and their teachers participated in a series of mathematical modelling activities from the 5th grade through to the 7th grade. The problems involve authentic situations that need to be interpreted and described in mathematical ways. We examine the mathematical understandings and mathematisation processes that the children used in constructing their models for the final problem (*Summer Reading*). We report on the nature of the problem factors that the children chose to consider, the operations they applied, the types of transformations they made through these operations, and the representations they used in documenting their models.

In a technology-based age of information, educational leaders from different walks of life are emphasizing a number of key understandings and abilities for success beyond school. These include the ability to make sense of complex systems through constructing, describing, explaining, manipulating, and predicting such systems (such as sophisticated buying, leasing, and loan plans); to work on multi-stage and multi-component projects in which planning, monitoring, and communication processes are critical for success; and to adapt rapidly to ever-evolving conceptual tools and resources (English, 2002a; Lesh & Doerr, 2003).

With the increasing availability of such tools, it is imperative that students be given experiences that encourage them to interpret mathematical situations in different ways and to communicate their understandings of these situations meaningfully to their peers. While technology can remove the computational complexity of mathematical problems, it does not remove the need for students to choose carefully the tool/s and resources to be used and to transform problem data into forms that can be handled effectively by these tools. The results obtained must be interpreted, documented, and communicated in forms that clearly and effectively convey the products of problem solution. One approach to providing students with these competencies is through mathematical modelling (English & Watters, 2005; Lesh & Doerr, 2003). Traditionally, mathematical modelling has been reserved for the secondary school years (e.g., Galbraith, Blum, Booker & Huntley, 1998), but recent research (e.g., English & Watters, 2005) has indicated that primary school children can participate successfully in meaningful modelling activities.

This paper addresses year 7 children's mathematical modelling at the end of a three-year program of model-eliciting, model-exploration, and model-application activities. We examine children's models for the final model-application problem, *Summer Reading*. We consider the nature of the problem factors children chose to work with, the operations they applied, the types of transformations they made through these operations, and the representations they used in documenting their model. We begin by reviewing the nature of mathematical modelling experiences for the primary school.

A modelling approach to the teaching and learning of mathematics focuses on the mathematisation of realistic situations that are meaningful to the learner. The emphasis on modelling involves three important shifts in the approach to teaching and learning mathematics, namely, in (1) the nature of the quantities and operations that are useful, (2) the use of contexts that will elicit the creation of useful systems (or models), and (3) the development and refinement of such models in ways that are generalisable (Doerr & English, 2003; Lesh & Doerr, 2003). We briefly discuss each of these aspects in turn.

The quantities and operations that are needed to mathematise realistic situations often go beyond what is traditionally taught in school mathematics. The types of quantities needed in realistic situations include accumulations, probabilities, frequencies, ranks, and vectors, while the operations needed include sorting, weighting, organizing, selecting, and transforming entire data sets rather than single, isolated data points (Doerr & English, 2001). In solving typical school "word problems," students generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. In most cases, the problem information has already been carefully mathematised for students. The students' goal is to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations.

In modelling tasks, however, students' goal is to make sense of the situation so that they can mathematise it in ways that are meaningful to them. This involves a cyclic process of interpreting the problem information, selecting relevant quantities, defining operations that may lead to new quantities, and creating meaningful representations (Lesh & Doerr, 2003). Oftentimes, the problem information might be incomplete, ambiguous, or undefined, and there might be too much or too little data; visual representations might also be difficult to interpret (as in real-world situations). When presented with information of this nature, children might make unwarranted assumptions or might impose inappropriate constraints on the products they are to develop.

A modelling approach to problem situations explicitly uses meaningful contexts that elicit the creation of useful systems or models. The modelling process begins with the elicitation stage, which confronts students with the need to develop a model to describe, explain, or predict the behaviour of some experienced system. Generalizing and re-using models are central activities in a modelling approach to learning mathematics. Hence, in each year of our program we designed sequences of activities that commenced with the children engaging with non-routine problem situations that elicited the development of significant mathematical constructs (model-eliciting problems). The children then extended, explored, and refined those constructs in other problem situations (model-exploration and model-application problems), leading to a generalisable system (or model) that could be used in a range of contexts. The problems provided us with a setting in which we could examine the development of the children's initial interpretations of the problem situation, their reasoning in selecting and working with quantities, operations, and representations, and their multiple cycles of review and modification. Because the problems are designed for small group work, we could observe the numerous questions, conjectures, conflicts, revisions, and resolutions that arose as the children developed, tested, and prepared to communicate their products.

Design and Methodology

In this study we worked with one class of children and their teachers from the fifth grade (10 years old) through to the end of the seventh grade (12 years old). We employed a longitudinal teaching experiment involving multilevel collaboration (English, 2003; Lesh & Kelly, 2000). At the first level, children engage in mathematical modelling, as discussed above. At the second and third levels, the classroom teacher works collaboratively with the researchers in designing and implementing the modelling problems. These problems also serve as challenging and thought-provoking experiences for the teachers as they explore the mathematical ideas being developed, consider appropriate implementation strategies, and promote learning communities within their classrooms. We confine our discussion here to the children's developments.

Modelling Activities and Procedures

In the first year of the program we implemented a number of preparatory modelling activities followed by a model-eliciting activity and a model-exploration activity. In each of the second and third years we implemented an initial model-eliciting problem, a model-exploration problem, and two model-application problems. The problems involved interpreting and dealing with multiple tables of data; creating, using, modifying, and transforming quantities; exploring relationships and trends; and representing findings in visual and text forms. The key mathematical ideas of rate and proportion, and ranks and weighted ranks were also included. The children had had no formal exposure to or instruction on these core mathematical ideas and processes prior to commencing the activities.

The activity addressed here (*Summer Reading Problem*) was the final problem the children completed. The students were to develop a fair rating system to award points to students participating in a summer reading program. The children were presented with a scenario (presented in the form of a newspaper article) about their city council library conducting an annual summer reading program with prizes to be won. The article explains that "*Students can choose from an approved collection of books that the library has placed on reserve. The books have been classified by grade level (according to difficulty of the book), to help the students choose which books to read. However, students may read any of the books, regardless of their current grade level.*" The children were given two tables of data: the first table comprised the titles of 14 books, their authors, the reading level (by grade from 4 to 10), and the number of pages. The second table listed the titles again plus a brief description of each book. Following a number of "readiness questions" about the article and the tables, the children worked the problem shown in Figure 1.

In term 4, the children worked this problem in small groups during two 50-minute sessions and presented their models to the class in a third session. We observed the children as they worked the problem and, where appropriate, asked the children to explain or justify a response. We did not give any explicit teaching. When the groups reported to the class, they explained and justified the models they had developed and then invited feedback from their peers. This group reporting was followed by a whole class discussion that compared the features of the mathematical models produced by the various groups.

Information: The Brisbane City Council Library and St. Peters School are sponsoring a summer reading program. Students in grades 6-9 will read books and prepare written reports about each book to collect points and win prizes. The winner in each class will be the student who has earned the most reading points. The overall winner will be the student who earns the most points. A collection of approved books has already been selected and put on reserve. See the previous page for a sample of this collection. Students who enrol in the program often read between ten and twenty books over the summer. The contest committee is trying to figure out a fair way to assign points to each student. Margaret Scott, the program director, said, "Whatever procedure is used, we want to take into account: (a) the number of books, (b) the variety of the books, (c) the difficulty of the books, (d) the lengths of the books, and (e) the quality of the written reports.
Note: The students are given grades of A+, A, A-, B+, B, B-, C+, C, C-, D, or F for the quality of their written reports.
Your mission: Write a letter to Margaret Scott explaining how to assign points to each student for all of the books that the student reads and writes about during the summer reading program.

Figure 1. Information and goal of the Summer Reading Problem.

Data collection and analysis

Data sources included audiotapes and videotapes of the children's group work and classroom presentations. Field notes, children's work sheets, and final reports detailing their models and how they developed them were also important data sources. The tapes were transcribed and, for the present paper, analysed for evidence of the mathematical understandings and mathematisation processes that the children used in building their models. To assist us in our analysis we used a modified version of Carmona's (2004) assessment tool for describing students' mathematical knowledge. We examined the nature of the problem factors that the children chose to consider, the operations they applied, the types of transformations they made through these operations, and the representations they used in documenting their final model.

Results

Prior to addressing the children's models, it is worth commenting that over the course of the three-year program, the children developed a number of core mathematical understandings and processes. As has been reported (e.g., English, 2002b; 2004), the children displayed cycles of increasing sophistication of mathematical thinking during the course of problem solution, with each cycle demonstrating a shift in thinking (Doerr & English, 2003). The significant social interactions that occurred in these cycles have also been documented (e.g., English, 2002b). For this paper we focus our attention on children's final models for the *Summer Reading Problem* with the aim of revealing their mathematical understandings and processes.

Table 1 displays children's model development and representational forms for each of five student groups. The *representational formats* included the use of tables, text, lists, and formulae. In building their model, children chose to work with some or all of the *problem factors*, namely, number of books read, variety of books considered, reading level of the books, length of the books, student's grade level, and quality of written reports. Children's *operations* on these factors included assigning value points, using interval quantities, using weighting, aggregating quantities, and using informal measures of rate. One group of children also imposed constraints on the use of their operations.

The ways in which the children worked with and operated on the problem factors can be seen in the children's written reports, which are reproduced next. In considering the problem factors, it was the variety of books that received the least attention. This is not surprising, given that a measure of variety requires a consideration of several factors. As one group explained, “With different variety of books—just say like if you read three different variety of books in subject or level or pages—just variety, so varieties include level, pages, and subject.” It is interesting to note that all but one group created formulae as part of their model, while only one group used any form of a table (which they used in developing their model but chose not to display in their report.)

The children displayed various *transformation processes* as they developed their models. Group 1 created new quantities (e.g., “Year 9 reads 4 books”) and transformed these into other quantities (e.g., “= 1 point”). Group 2 quantified selected problem factors (e.g., report quality) and also transformed quantities into other quantities (through aggregation). Group 3 quantified selected factors (grade level, book reading level). They also transformed quantities into other quantities when they considered the difference between the students’ grade level and the book grade level, and then determined if that difference was a multiple of the student’s grade level. In the comprehensive model produced by group 4, the children quantified several problem factors and also quantified the diversity of book reading. Finally, group 5 quantified selected factors (grade level, book reading level, report quality).

Table 1
Children’s Model Development and Representational Forms for each of the Five Student Groups

Group	Representation Format					Development of Model										
						Factors						Operations				
	Table	Text	List	Computation	Formulae	No. of books	Variety	Length	Student's grade level	Book's reading level	Report's quality	Assigning value points	Using interval quantities	Using weighting	Aggregating quantities	Informal rate
1		√	√		√	√			√			√	√			
2		√	√	√	√			√		√	√	√	√	√	√	
3		√	√		√				√	√		√				
4		√			√	√	√	√	√	√	√	√	√	√		√
5	√	√							√	√	√	√				√

Group 1’s Model

Dear Margaret Scott
We have found a solution to the problem you have given us. Our information is below.
Year 9 reads 4 books = 1 point 8=4=2
Year 9 reads 5 books = 2 points 8=5=3
Year 9 reads 6 books = 3 points 8=6=4

Year 9 reads 7 books = 4 points
9=8=5
9=9=6
9=10=7

8=7=5
8=8=6
8=9=7
8=10=8

(The group used the same pattern as was used for the grade 7 and grade 6 reading books.)

Group 2's Model

Dear Mrs Scott
We have found a solution to your problem.
We think that the grade of the book times by 10 and then add the amount of pages to that. After doing this you should look at the book report grade and if it is between an A+ to an A- they get another 50 points, B+ to a B- 40 points, C+ to a C- 30 points, D+ to a D- 10 points and no points for an F. The three scores should be added together and that will be the score of the book. We chose this way because the person involved will easily get a high score. There for lifting their high esteem and they will be encouraged to read more which is the whole point of this activity. P.S the total score of all the books read should be added up in the end and that will be the total score of all of the books.

- The grade of the book x 10
- Plus the amount of pages in it
- Add the grade of the book report

A+ to A− = 50 points
B+ to B− = 40 points
C+ to C− = 30 points
D+ to D− = 20 points
E+ to E− = 10 points
F = 0 points

Group 3's Model

We have resolved the problem about the points issue. We have come up with a fair point system. We surround it by the level you read at. 2 points if you read at your level, 3 points above your level 2 levels higher. You get four points if it is 4 levels above your level. You get one point if it is one below your level. You get zero points if it is two levels below you. We think this is the easiest way to reward the children.
2 = year level
3 = above your level twice
4 = above your level 4 times
1 = below your level one times
0 = below your level two times

Group 4's Model

Dear Miss Margaret Scott,
We have found a solution to your problem. But first we will tell you our strategy. For every book a student reads they would get 1 bonus point per book. Next if they read three different grades of books or more they would get 5 points. Eg. If a grade 4 girl/boy reads a grade 4, 5 and 6 book then he/she would get 5 points. We gave points for the difficulty of books by if you were in grade 6 and you read a grade 6 then you would get 3 points because you half the grade. Although there is a rule of you can only read two levels below

your grade to receive point and as many levels above that you are capable of. If you were in grade 5 and your read a grade 4 book you would still get 4 points because you are still halving it. But if you were in grade 9 and you read a grade 5 book you would not get any points. Because the book is too easy. We went by a code for the lengths of books: 50-70 = 3 points, 71-100 = 4 points, 101-170 = 5 points, 171-220 = 6 points and 221 and up = 7 points. We also made a code for the written reports as well which is: F=0, D=1, C- =2, C=3, C+ = 4, B- = 5, B=6, B+ = 7, A- = 8, A=9 and A+ = 10. we hope this helps you decide on how to give out points and awards during the summer.

Group 5's Model

Dear Margaret Scott
We have formulated a point system to determine results for your reading marathon. The system works on a point basis. If you read a book that is based for your grade level you receive 10 points although if you read a book higher than your grade level you receive 2 points for every grade level you read up and you 2 points go down for every grade level lower. If you get an F for the report you don't get bonus points but from every grade that goes up from F you receive one point.

Discussion and Concluding Points

We have examined the mathematical models constructed by a class of year 7 children at the end of a three-year modelling program. Our focus here has been on the children's documented models for the *Summer Reading Problem*, where they were to develop a fair rating system for awarding points to students participating in a summer reading program. We have addressed the problem factors that the children considered, the operations they applied, the nature of the transformations they made through these operations, and the representations they used in documenting their model.

The different models the children generated revealed a range of mathematical understandings and processes that they, themselves, had developed over the course of working the modelling problems. For example, the children debated which factors to include in their models and how to quantify them. In quantifying their data they assigned value points, used interval quantities, weighted some factors, aggregated quantities, and applied informal measures of rate. In so doing, the children created new quantities, transformed problem factors into quantities, transformed quantities into other quantities, and quantified a measure of book variety/diversity. Finally, they created mostly formulae and lists to support their textual accounts of their model construction.

While not documented here, the transcripts also showed how the children focused on building their models from the outset, in contrast to previous years in which they tended to get bogged down initially in debating irrelevant contextual issues. The year 7 children also made more use of examples and counter-examples in justifying and questioning the ideas they proposed. For example, group 4 commenced the *Summer Reading Problem* with a child commenting, "Well, let's go over the possibilities like the number of books they read… Well, the number of books they read is up to them and what grades they do. So there should be like one point of order for each book they read – like, one bonus point. And depending on the grade, if they read a grade 4 book they get 8 points – like, you double it." Another group member responded, "But it depends. If they're in grade 10 and they read a grade 4 book, then they wouldn't really get any points for that." Issues pertaining to an older student reading a grade 4 book were also debated by all groups on presenting their models to the class. Points of clarification, justification, and argumentation dominated the discussion. For example, Anne argued, “If a grade 9 reads a grade 4 book, the majority of grade 9s would be able to do a report better than a grade fourer. And so therefore, they’d get more points.” Jack disagreed and explained, “Yeah, but it is all graded for each grade. They’re not comparing against each other; they’re writing like a book report for each grade.

In sum, the children's independent mathematical growth, along with the development of their collaborative problem-solving skills, over the three-year program has provided strong support for the inclusion of modelling in the primary mathematics curriculum.

References

Carmona, G. (2004). *Designing an assessment tool to describe students’ mathematical knowledge*. Ph.D. Thesis. West Lafayette, IN: Purdue University

Doerr, H., & English, L. D. (2001). A modelling perspective on students' learning through data analysis. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (pp. 361-368). Utrecht University: PME.

Doerr, H., & English, L. D. (2003). A modelling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education,34*(2), 110-136.

English, L. D. (2002a). Promoting learning access to powerful mathematics for a knowledge-based era. In D. Edge & Y. Ban Har (Eds.), *Mathematics education for a knowledge-based era* (vol. 1, pp. 100-107). National Institute of Education, Singapore: Association of Mathematics Educators.

English, L. D. (2002b). Development of 10-year-olds’ mathematical modelling. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th International PME Conference* (pp. 329-336). Norwich: PME.

English, L. D. (2003). Reconciling theory, research, and practice: A models and modelling perspective. *Educational Studies in Mathematics,* 54(2 & 3), 225-248.

English, L. D. (2004). Mathematical modelling in the primary school. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: towards 2010* (pp. 207-214). Sydney: MERGA.

English, L. D., & Lesh. R. A. (2003). Ends-in-view problems. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on mathematics problem solving, learning, and teaching* (pp. 297-316). Mahwah, NJ: Lawrence Erlbaum.

English, L. D., & Watters, J. J. (2005). Mathematical modelling in third-grade classrooms. *Mathematics Education Research Journal, 16*(3), 59-80.

Galbraith, P. L., Blum, W., Booker, G., & Huntley, I. D. (1998). *Mathematical modelling: Teaching and assessment in a technology-rich world*. West Sussex: Horwood Publishing Ltd.

Lesh, R. A., & Doerr, H. (2003). Foundations of a Models & Modeling Perspective on Mathematics Teaching and Learning. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modelling perspective on mathematics teaching, learning, and problem solving* (pp. 3-34). Mahwah, NJ: Erlbaum.

Lesh, R. A., & Kelly, A. E. (2000). Multi-tiered teaching experiments. In R. A. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Mahwah, NJ: Erlbaum

Lesh, R., & Yoon, C. (2004). What's Distinctive in (Our Views about) Models & Modeling Perspectives on Mathematics Problem Solving, Learning and Teaching? In H. Henn & W. Blum (Eds.), *ICMI Study 14: Applications and Modeling in Mathematics Education*. Pre-Conference Volume.